10 Podgrupe edinke. Kvocientne grupe.

Definicija (edinka)
Naj bo G grupa in $H \leq G$. Podgrupi H pravimo edinka, če je $aH = Ha$ za vsak $a \in G$. To označujemo z $H \lhd G$.

1. Naj bo G Abelska grupa. Pokaži, da je vsaka podgrupa H grupe G edinka.

2. (a) Naj bo $H = \{(1), (12)\}$. Ali je H edinka podgrupe S_3?

(b) Naj bo $N = \{(1), (123), (132)\}$. Ali je $N \lhd S_3$?

Izrek (test za edinke)
Podgrupa H grupe G je edinka v grupi G če in samo če $xHx^{-1} \subseteq H$ za vsak $x \in G$.

3. Dokaži izrek zgoraj.

4. (a) Naj bo $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \text{ in } ac \neq 0 \right\}$. Ali je $H \lhd \mathrm{GL}_2(\mathbb{R})$? Obrazloži svojo trditev.

(b) Pokaži da je $\mathrm{SL}_2(\mathbb{R}) \lhd \mathrm{GL}_2(\mathbb{R})$.

5. (a) Če je $[G : H] = 2$ ($[G : H]$ je indeks podgrupe H v grupi G) pokaži da je potem $H \lhd G$.

(b) Pokaži da je A_n edinka grupe S_n.

6. Naj bosta H in J edinki grupe G. Če je $H \cap J = \{e\}$ (e je identiteta) pokaži, da je potem $xy = yx$ za vse $x \in H$, $y \in J$.

7. Naj bo $H \lhd G$, in naj bo $K \leq G$. Definirajmo

$$HK = \{hk : h \in H, k \in K\}.$$

Pokaži da je $HK \leq G$.

8. Naj bo G grupa in naj bo N podgrupa grupe G. Pokaži da je N edinka grupe G če in samo če za $\forall g \in G$ velja, da je $gNg^{-1} = N$.

Izrek (kvocientne grupe)
Naj bo G grupa in naj bo H edinka grupe G. Množica $G/H = \{aH \mid a \in G\}$ je grupa, glede na operacijo $(aH)(bH) = abH$, reda $[G : H]$.

9. Dokaži izrek zgoraj.

10. (a) Pokaži, da je kvocientna grupa ciklične grupe ciklična.

(b) Pokaži, da je kvocientna grupa abelske grupe abelska.

11. Naj bo $H = \langle 4 \rangle$ podgrupa grupe $G = \mathbb{Z}$ (generirane s številom 4).

(a) Napiši vse elemente grupe $H = \langle 4 \rangle$.

(b) Pokaži, da je H edinka grupe G.

(c) Napiši vse elemente kvocientne grupe $G/H = \mathbb{Z}/\langle 4 \rangle$.

(d) Napiši Cayley-evo tabelo za $\mathbb{Z}/\langle 4 \rangle$.

(e) Določi red elementa $2 + \langle 4 \rangle$ v grupi $\mathbb{Z}/\langle 4 \rangle$.

12. Določi red elementa $2 + \langle 5 \rangle$ v grupi $\mathbb{Z}/\langle 5 \rangle$.

13. Naj bo $H = \langle 6 \rangle$ podgrupa grupe $G = \mathbb{Z}_{18}$.

(a) Napiši vse elemente grupe $H = \langle 6 \rangle$.

(b) Pokaži, da je H edinka grupe G.

(c) Napiši vse elemente kvocientne grupe $\mathbb{Z}_{18}/\langle 6 \rangle$.

(d) Napiši Cayley-evo tabelo za $\mathbb{Z}_{18}/\langle 6 \rangle$.

(e) Določi red elementov $2 + \langle 6 \rangle$, $3 + \langle 6 \rangle$ in $5 + \langle 6 \rangle$ v grupi $\mathbb{Z}_{18}/\langle 6 \rangle$.

14. Naj bo $K = \langle 15 \rangle$ podgrupa grupe $G = \mathbb{Z}$ (generirane s številom 15).

(a) Napiši vse elemente grupe $K = \langle 15 \rangle$.

(b) Pokaži, da je K edinka grupe G.

(c) Napiši vse elemente kvocientne grupe $G/K = \mathbb{Z}/\langle 15\rangle$.

(d) Določi red elementov $3+\langle 15\rangle$, $4+\langle 15\rangle$, $5+\langle 15\rangle$ in $6+\langle 15\rangle$ v grupi $\mathbb{Z}/\langle 15\rangle$.

(e) Pokaži, da je $G/K = \mathbb{Z}/\langle 15\rangle$ ciklična.

(f) Pokaži da je $\mathbb{Z}/\langle 15\rangle$ izomorfna grupi $\mathbb{Z}_{15}$.

15. Naj bosta $G = \langle 6\rangle$ in $H = \langle 24\rangle$ podgrupi grupe $\mathbb{Z}$.

(a) Pokaži, da je H edinka v grupi G. Napiši odseke podgrupe H v grupi G. Napiši Cayley-evo tabelo za G/H.

(b) Pokaži, da je $G/H = \langle 6\rangle/\langle 24\rangle$ izomorfna grupi $\mathbb{Z}_4$.

16. Naj bo $G = U(16)$ grupa vseh pozitivnih celih števil manjših od 16, ki so tuja s 16, glede na operacijo množenja modulo 16.

(a) Kakšen je red grupe G?

(b) Kakšen je red elementa $15 \in U(16)$?

(c) Naj bo $H = \langle 15\rangle$ podgrupa grupe $U(16)$ (generirana s številom 15). Določi red kvocientne grupe $U(16)/\langle 15\rangle$.

(d) Napiši Cayley-evo tabelo za $U(16)/H$.

17. Naj bo $G = H \times K$ (kje sta H in K dani grupi). Pokaži, da je potem $H \times \{e\} \lhd G$.

18. Poišči red dane kvocientne grupe

(a) $(\mathbb{Z}_4 \times \mathbb{Z}_4)/(\langle 2\rangle \times \langle 2\rangle)$;

(b) $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$.

19. Določi red elementa $3\langle 16\rangle$ v grupi $U(35)/\langle 16\rangle$.

POMEMBNI REZULTATI (Podgrupe edinke. Kvocientne grupe.)

1. **(Test za edinke.)** Podgrupa H grupe G je edinka v grupi G če in samo če $xHx^{-1} \subseteq H$ za vsak $x \in G$.
2. **(Kvocientne grupe.)** Naj bo G grupa in naj bo H edinka grupe G. Množica $G/H = \{aH \mid a \in G\}$ je grupa, glede na operacijo $(aH)(bH) = abH$, reda $[G : H]$.
3. **(G/Z izrek)** Naj bo G grupa in naj bo $Z(G)$ center grupe G. Če je $G/Z(G)$ ciklična grupa, potem je G abelska.
4. ($G/Z(G) \cong \mathrm{Inn}(G)$) Za vsako grupo G, je $G/Z(G)$ izomorfna z $\mathrm{Inn}(G)$.
5. **(Cauchijev izrek za abelske grupe.)** Naj bo G končna grupa in naj p deli $|G|$, kje je p praštevilo. Potem obstaja element $a \in G$ ($a \neq e$) t.d. $a^p = e$ (obstaja element reda p).

Rešitve: **1.** $[h \in H, \forall g \in G\ gh = hg \Rightarrow gH = \{gh \mid h \in H\} = \{hg \mid h \in H\} = Hg$, $gH = Hg\ \forall g \in G$, $H \lhd G]$ **2.**(a) $[(123)H = \{(123),(13)\}$, $H(123) = \{(123),(23)\}$, $H \ntriangleleft S_3]$. (b) $[(12)N = N(12) = \{(12),(13),(23)\}$, $N \lhd S_3]$ **3.** $H \lhd G \Rightarrow \forall x \in G\ \forall h \in H\ \exists h' \in H$ t.d. $xh = h'x \Rightarrow xhx^{-1} = h' \Rightarrow xHx^{-1} \subseteq H$... **4.**(a) $[A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $A^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $A = A^{-1}$, $A \in \mathrm{GL}_2(\mathbb{R})$, $B = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $B \in H$, $ABA^{-1} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \notin H$, $H \ntriangleleft \mathrm{GL}_2(\mathbb{R})]$ **5.**(b) $[|S_n| = n!$, $|A_n| = n!/2$, #(levih odsekov grupe A_n v $S_n) = [S_n : A_n] = 2$, #(desnih odsekov grupe A_n v $S_n) = 2$, če je $a \in A_n$ potem $aA_n = A_n = A_na$, če $a \notin A_n$ potem $aA_n \neq A_n \neq A_na$, $S_n = A_n \cup aA_n = A_n \cup A_na$...] **6.** $[x \in H$, $y \in J$, $xyx^{-1}y^{-1} \in G$, $xyx^{-1} \in J$, $y^{-1} \in J$, $xyx^{-1}y^{-1} \in J$...] **7.** $[e = ee \in HK$, $\forall a = h_1k_1$ in $\forall b = h_2k_2$ kjer $h_1, h_2 \in H$ in $k_1, k_2 \in K$ $\exists h' \in H$ t.d. $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1(k_1k_2^{-1})h_2^{-1} = (h_1h')(k_1k_2^{-1})$ (npr. $h' = (k_1k_2^{-1})h_2^{-1}(k_1k_2^{-1})^{-1}) \Rightarrow ab^{-1} \in HK]$ **8.**

Richard Dedekind

> Richard Dedekind was not only a mathematician, but one of the wholly great in the history of mathematics, now and in the past, the last hero of a great epoch, the last pupil of Gauss, for four decades himself a classic, from whose works not only we, but our teachers and the teachers of our teachers, have drawn.
>
> *Edmund Landau,*
> *Commemorative Address,*
> *to the Royal Society of Göttingen,*

Richard Dedekind was born on October 6, 1831, in Brunswick, Germany, the birthplace of Gauss. Dedekind was the youngest of four children of a law professor. His early interests were in chemistry and physics, but he obtained a doctor's degree in mathematics at the age of 21 under Gauss at the University of Göttingen. Dedekind continued his studies at Göttingen for a few years, and in 1854 he began to lecture there.

Dedekind spent the years 1858–1862 as a professor in Zürich. Then he accepted a position at an institute in Brunswick where he had once been a student. Although this school was less than university level, Dedekind remained there for the next 50 years. He died in Brunswick in 1916.

During his career, Dedekind made numerous fundamental contributions to mathematics. His treatment of irrational numbers, "Dedekind cuts," put analysis on a firm, logical foundation. His work on unique factorization led to the modern theory of algebraic numbers. He was a pioneer in the theory of rings and fields. The notion of ideals as well as the term itself are attributed to Dedekind. Mathematics historian Morris Kline has called him "the effective founder of abstract algebra."

On the Google Drive please find solutions for the following problems:

1. Prove or disprove that H is a normal subgroup of G. (a) $G = U(720)$, $H = \langle 49 \rangle$. (b) $G = S_4$, $H = \langle (1234) \rangle$. (c) $G = D_{2n}$, $H = \langle R_{180} \rangle$. **2.** Let $G = U(16)$ and $H = \langle 9 \rangle$. Compute the Cayley table of G/H. **3.** Let G be a group and H be a subgroup of index 2. Prove that $H \lhd G$. **4.** (a) Suppose that $H \leq G$. Then for every $g \in G$, show that $gHg^{-1} \leq G$. (b) For some positive integer k, let H be a unique subgroup of G of order k. Prove that $H \lhd G$. (c) Let G be a group of order 245 and H be a subgroup of order 49. By using (b), show that $H \lhd G$. **5.** By using internal direct products, show that $D_6 \cong D_3 \times \mathbb{Z}_2$. **6.** Let G be a group and H, K be two subgroups of G. Recall that $HK = \{hk \mid h \in H,\ k \in K\}$. Show that if $H \lhd G$, then $HK \leq G$. **7.** Let $H = \{(1), (12)\}$. Is H normal in S_3? **8.** Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3 \rangle / \langle 12 \rangle$ is isomorphic to $\mathbb{Z}_4$. Similarly, prove that $\langle 8 \rangle / \langle 48 \rangle$ is isomorphic to $\mathbb{Z}_6$. Generalize to arbitrary integers k and n. **9.** Let $G = \mathbb{Z}_4 \times U(4)$, $H = \langle (2,3) \rangle$, and $K = \langle (2,1) \rangle$. Show that G/H is not isomorphic to G/K. (This shows that $H \cong K$ does not imply that $G/H \cong G/K$.) **10.** Prove that a factor group of a cyclic group is cyclic. **11.** What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$? **12.** Prove that an Abelian group of order 33 is cyclic. **13.** Determine the order of $(\mathbb{Z} \times \mathbb{Z})/\langle (2,2) \rangle$. Is the group cyclic? **14.** The group $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/\langle (2,2) \rangle$ is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Determine which one by elimination. **15.** Prove that D_4 cannot be expressed as an internal direct product of two proper subgroups. **16.** In $\mathbb{Z}$, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = HK$. Does $\mathbb{Z} = H \times K$? **17.** If H is a normal subgroup of a group G, prove that $C(H)$, the centralizer of H in G, is a normal subgroup of G. **18.** Let N be a normal subgroup of G and let H be a subgroup of G. If N is a subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G. **19.** If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

Computer Tutorial 10.[27][28]

vector space

<table>
<tr><th>Input</th><th>Meaning</th></tr>
<tr><td><code>R := RealField();
V := VectorSpace(R,4);
a := V![4,3,2,1];
b := V![1,-2,0,-3];</code></td><td>Setting up the vector space. Use MAGMA to create a vector space $\mathcal{V}$ of dimension 4 over the real numbers $\mathbb{R}$. Then define two vectors $(4,3,2,1)^\top$ and $(1,-2,0,-3)^\top$ named a and b.</td></tr>
<tr><td><code>a,b;

V;</code></td><td>Displaying values. To see the value of the quantities you have defined, you can use the <code>print</code> command. In fact the word <code>print</code> can be omitted if you wish.</td></tr>
<tr><td><code>InnerProduct(a,b);
Sqrt(InnerProduct(a,a));</code></td><td>Inner products and length.</td></tr>
<tr><td><code>Length := func< v | Sqrt(InnerProduct(v,v)) >;
Angle := func< u,v | Arccos(InnerProduct(u,v) /
(Length(u) * Length(v))) >;
print Angle(a,b);

asDegree := func< x | x * 180/Pi(R) >;
print asDegree(Angle(a,b));</code></td><td>Angles. You should be able to find the angle between a and b using the commands from above together with the <code>Arccos</code> function. You will see that this involves quite a lot of typing and very often you will make typing mistakes. To make things easier, first define an abbreviation for the <code>Length</code> function and then define the <code>Angle</code> function. Now you can print the angle between <code>a</code> and <code>b</code>. (Notice that <code>*</code> is used for multiplication.) The answer will be in radians. But what if you want the answer in degrees? Notice that in MAGMA you use <code>Pi(R)</code> to get the number π. (The <code>R</code> in this refers to the real field $\mathbb{R}$ (to which π belongs). MAGMA always needs to be told which set contains each object.)</td></tr>
<tr><td><code>Distance := func< u,v | Length(u-v) >;
Distance(a,b);</code></td><td>Define a MAGMA function <code>Distance</code> such that <code>Distance(a,b)</code> gives the distance from <code>a</code> to <code>b</code> .</td></tr>
<tr><td><code>u := V![1,0,-5,7];
v := V![21,2,2,-2];
print Length(u);
print Length(v);
print Angle(u,v);
print asDegree(Angle(u,v));</code></td><td>Let $u = (1,0,-5,7)$ and $v = (21,2,2,-2)$. Find the lengths of u and v and the angle between them.</td></tr>
<tr><td><code>W := VectorSpace(R,3);
u1 := W![0,0,0]; u2 := W![1,1,0];
u3 := W![1,0,1]; u4 := W![0,1,1];
print Distance(u1,u2); print Distance(u1,u3);
... print Distance(u3,u4);
c := W![1/2,1/2,1/2];
r1 := u1 - c; r2 := u2 - c;
r3 := u3 - c; r4 := u4 - c;
print Angle(r1,r2); print Angle(r1,r3);
... print Angle(r3,r4);
print asDegree(Angle(r1,r2));
print asDegree(Angle(r1,r3));
... print asDegree(Angle(r3,r4));
quit;</code></td><td>The four points $(0,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are the vertices of a tetrahedron in $\mathbb{R}^3$. (i) We show that all six edges of this tetrahedron have the same length. (ii) Given that the centre of this tetrahedron is $(1/2,1/2,1/2)$, we calculate the angle between two rays joining the centre to two of the vertices. We check that we get the same answer whichever two vertices you choose. (This tetrahedron can be seen as a model of a methane molecule, with a carbon atom at the centre and hydrogen atoms at the vertices. The angle in Part (ii) is the “bond angle”.)</td></tr>
</table>

[27] To write MAGMA code please open: `http://magma.maths.usyd.edu.au/calc/`
[28] See also: `http://www.maths.usyd.edu.au/u/bobh/UoS/MATH2008/ctut10.pdf`